ON CARDINALITY, COHOMOLOGY AND A CONJECTURE OF ROSENBERG AND ZELINSKY

BY
ROBERT E. MACRAE [1]

1. **Introduction.** It might be said that the cohomology of associative algebras (for an exposition see [1, Chapter IX]) first became of real interest when Hochschild showed [2, Theorem 4.1] that, for algebras of finite order over a field, the identical vanishing of the first cohomology group is equivalent to the classical notion of separability for such algebras. For commutative algebras of finite order over a field, this theorem had been shown some years earlier by E. Noether in the posthumous [4]. Then in 1956 Rosenberg and Zelinsky [5, Theorem 1] showed the surprising fact that if S is an associative algebra over a field K and the first cohomology group of S vanishes identically, then S is necessarily of finite order over K. They then went on to show that if S is locally separable and of countable order over K, then the second cohomology group of S vanishes identically [5, Theorem 4]. Zelinsky had already [6, p. 316] given an example to show that the countability hypothesis cannot be dropped. It was natural, therefore, for them to conjecture [5, bottom of p. 86] that the identical vanishing of the second cohomology group suffices to force countable order over the ground field—at least when S is a field. By utilizing Kaplansky's remarkable piece of universal algebra [3], we are now able to give an affirmative answer to the conjecture of Rosenberg and Zelinsky—at least when S is a field. For more general algebras the problem remains open.

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2. **Results.** If S is a K-algebra, then $\dim_K(S)$ will denote the Hochschild dimension of S. If S is a ring, then $\mathrm{gl.dh}(S)$ will denote the global dimension of S. See [1] for further details.

LEMMA 2.1. *Let S be a commutative ring. If* $\mathrm{gl.dh}(S) \leqq 2$, *then S possesses no nontrivial nilpotent elements.*

Proof. Let N be the ideal of S consisting of all the nilpotent elements of S. We wish to show $N = 0$. Let P be any maximal ideal of S and denote the canonical extension of N to the ring of quotients, S_P, by N_P. One knows that N_P is the

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ideal of nilpotent elements in S_P and $N=0$ if and only if $N_P=0$ for all maximal ideals, P, in S. Furthermore, it is well known that $\text{gl.dh}(S_P) \leqq \text{gl.dh}(S)$. It suffices, therefore, to assume that S has precisely one maximal ideal. In this case we will show the stronger fact that S possesses no divisors of zero. Indeed, let $\mathscr{A}$ be a zero divisor and let A be its annihilator. One has the following standard exact sequences:

$$0 \to A \to S \to \mathscr{A}S \to 0,$$

$$0 \to \mathscr{A}S \to S \to S/\mathscr{A}S \to 0.$$

Since $\text{gl.dh}(S) \leqq 2$, we see that A is a projective S-module. Hence, by Kaplansky's result [3, Theorem 2], A is free. Since it is an ideal in a commutative ring, A must then be a principal ideal generated by a nondivisor of zero. This is, however, a contradiction since $\mathscr{A}A=0$.

COROLLARY 2.2. *Let K be a field and let S be a commutative, algebraic K-algebra. If* $\text{gl.dh}(S) \leqq 2$, *then S is a von Neumann ring (i.e., $\mathscr{A} \in \mathscr{A}^2S$ for all $\mathscr{A} \in S$).*

Proof. Let P be a prime ideal of S. One knows that S_P is again algebraic over K. Hence the maximal ideal of S_P consists only of nilpotent elements ($X^n=0$ for some n is the only possible equation over K). By Lemma 2.1, S_P is a field. Let $\mathscr{A} \in S$ and let $\bar{S}=S/\mathscr{A}^2S$. From the above, $\bar{S}_{\bar{P}}$ is a field for all prime ideals, $\bar{P}$ of $\bar{S}$. Hence $\bar{S}$ has no nilpotent elements aside from zero so $\mathscr{A} \in \mathscr{A}^2S$.

LEMMA 2.3. *Let K be a field and let S be a commutative, algebraic K-algebra. If* $\dim_K(S) \leqq 1$, *then* $\dim_K(S/A) \leqq 1$ *for all proper ideals, A, of S.*

Proof. By [1, Chapter IX, Proposition 7.4], $\dim_K(S \otimes_K S) \leqq \dim_K(S)+\dim_K(S)$. By [1, Chapter IX, Proposition 7.6], $\text{gl.dh}(S \otimes_K S) \leqq \dim_K(S \otimes_K S)$. Hence we have $\text{gl.dh}(S \otimes_K S) \leqq 2$. By Corollary 2.2, $S \otimes_K S$ is a von Neumann ring. By [3, Theorem 4], any projective ideal of $S \otimes_K S$ is generated by orthogonal idempotents. Consider the standard exact sequence

$$0 \to J \to S \otimes_K S \to S \to 0.$$

Since $\dim_K(S) \leqq 1$, we see that J is $(S \otimes_K S)$-projective and so is generated by orthogonal idempotents. Let A be any proper ideal of S, and let $\bar{S}=S/A$. We observe the following commutative, exact diagram:

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r] \arrow[d] & S \otimes_K S \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bar{J} \arrow[r] \arrow[d] & \bar{S} \otimes_K \bar{S} \arrow[r] \arrow[d] & \bar{S} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Since J is generated by orthogonal idempotents so is $\bar{J}$ and thus $\bar{J}$ is $(\bar{S}\otimes_K\bar{S})$-projective. In other words, $\dim_K(\bar{S}) \leqq 1$.

Our main result is now within reach.

THEOREM 2.4. *Let K be a field and let L be an extension field. Then* $\dim_K(L) = 1$ *if and only if* (i) *L is of countable* (*but not finite*) *order over K and* (ii) *L is a separable algebraic extension of K, or L is a finite, separable extension of an intermediate field, $K(\mathscr{A})$, generated by a single transcendental element.*

Proof. By the results of Rosenberg and Zelinsky [5] it suffices to show that if $\dim_K(L) = 1$ and L is algebraic over K then L is separable over K and of countable but not finite order. We show first that L is a separable extension of K. Let $\mathscr{A} \in L$. It is well known that $\mathscr{A}$ is separable if and only if $K[\mathscr{A}]\otimes_K K[\mathscr{A}]$ contains no nilpotent elements. Since K is a field the injection,

$$K[\mathscr{A}]\otimes_K K[\mathscr{A}] \to L\otimes_K L$$

is a monomorphism. Hence it suffices to show that $L\otimes_K L$ has no nilpotent elements. By [1, Chapter IX, Proposition 7.1], $\dim_L(L\otimes_K L) = \dim_K(L)$. By [1, Chapter IX, Proposition 7.6], $\mathrm{gl.dh}(L\otimes_K L) \leqq \dim_L(L\otimes_K L)$. Hence $\mathrm{gl.dh}(L\otimes_K L) \leqq 1$. By Lemma 2.1, $L\otimes_K L$ has no nilpotent elements. In order to show that the order of L over K is countable we first note that the classical Hochschild Theorem [1, Chapter IX, Theorem 7.10] assures us that the order cannot be finite. Consider now the standard exact sequence

$$0 \to J \to L\otimes_K L \to L \to 0.$$

We have already shown above that $\mathrm{gl.dh}(L\otimes_K L) \leqq 1$. Hence by Corollary 2.2 and [3, Theorem 4], J is generated by orthogonal idempotents. We claim first that these idempotent generators form a countable set. Indeed, let M be the algebraic closure of K and consider the following exact sequences

$$\begin{array}{ccccccccc}
0 \to & J\otimes_L M & \to & (L\otimes_K L)\otimes_L M & \to & L\otimes_L M & \to 0 \\
 & \| & & \| & & \| & \\
0 \to & J' & \longrightarrow & L\otimes_K M & \longrightarrow & M & \longrightarrow 0.
\end{array}$$

One observes that J' is a maximal ideal of $L\otimes_K M$ generated by orthogonal idempotents. Furthermore, by [1, Chapter IX, Proposition 7.1], $\dim_M(L\otimes_K M) = \dim_K L = 1$. Let $S = L\otimes_K M$, let $u_1, \cdots$ be orthogonal idempotents generating the maximal ideal J' and let $A_1, \cdots$ be ideals of S maximal with respect to the two properties (i) $A_i \leqq u_i S$ and (ii) $u_i \notin A_i$. Let $A = \sum A_i$. By Lemma 2.3, $\dim_M(S/A) \leqq 1$. One observes that the canonical images of $u_1, \cdots$ in S/A are orthogonal, primitive idempotents generating a maximal ideal of S/A. Since M is algebraically closed, S/A is isomorphic to a direct sum of copies of M (one for each u_i) with an identity element adjoined. It is, however, known [5, last paragraph of p. 86] that, for such an algebra, $\dim_M(S/A) \leqq 1$ implies that the set of generating idempotents is, at most, countable. Since the cardinality of the set of $u_1, \cdots$ is clearly the same as that of the set of orthogonal idempotents generating J, we have made good our claim. It now remains only to prove that L possesses a countable K-basis. Let $s_1, \cdots$ be a K-basis for L and let $v_1, \cdots$ be a (necessarily countable) set of orthogonal idempotents generating J. We have the equations

$$v_i = \sum \lambda_{ijk} s_j \otimes s_k,$$

where the coefficients are in K and each sum is finite. The subset of the $s_1, \cdots$ involved in the expression of the $v_1, \cdots$ is clearly countable. Let N be the intermediate field generated by this subset. Clearly N is of countable order over K. Consider the exact commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r] \arrow[d] & L \otimes_K L \arrow[r] \arrow[d] & L \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & J' \arrow[r] \arrow[d] & L \otimes_N L \arrow[r] \arrow[d] & L \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

It is clear from our construction that $J' = 0$. But this can only happen when $N = L$.

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COLUMBIA UNIVERSITY
NEW YORK, NEW YORK